Exam to the Lecture
Traffic Dynamics and Simulation
SS 2026
Solutions

Total 120 points

Problem 1 (30 points)

(a) The triangular fundamental diagram (per lane) is given by

$$Q(\rho) = \min\left[V_0\rho, \frac{1}{T}(1 - l_{\text{eff}}\rho)\right]$$

leading to a maximum

$$Q_{\max}(V_0, T, l_{\text{eff}}) = V_0\rho_c = \frac{V_0}{V_0T + l_{\text{eff}}}.$$

Consequently, the capacities are given as follows

- Outside of the bottleneck:

$$V_{01} = 120/3.6\,\text{m/s}, \quad T_1 = 1.2\,\text{s}, \quad l_{\text{eff}} = 8\,\text{m}, \quad K_1 = 2Q_{\max,1} = 1.389\,\text{veh/s} = 5\,000\,\text{veh/h}$$

- Inside the bottleneck:

$$V_{02} = 60/3.6\,\text{m/s}, \quad T_2 = 1.4\,\text{s}, \quad l_{\text{eff}} = 8\,\text{m}, \quad K_2 = 2Q_{\max,2} = 1.064\,\text{veh/s} = 3\,830\,\text{veh/h}$$

A demand of $3\,600\,\text{veh/h} < K_2$ can be accommodated by the bottleneck and no breakdown occurs.

(b) Since a demand of $4\,500\,\text{veh/h}$ exceeds K_2 but not K_1, the full demand reaches the bottleneck where a breakdown is occurs at $t = 0$ right the beginning of the bottleneck.

(c) Inversion of the fundamental diagram is best done on a per-lane basis, so the flow is given by half the bottleneck capacity (in congested regions, information propagates upstream, so the bottleneck, not the demand, determines the flow):

$$Q_{\text{cong}} = K_2/2 = 0.532\,\text{veh/s/lane}.$$

Inversion of the congested branch of the FD *outside of the bottleneck* gives

$$\rho_{\text{cong}} = \rho_{\max}(1 - Q_{\text{cong}}T_1) = \frac{1 - Q_{\text{cong}}T_1}{l_{\text{eff}}} = 45.2\,\text{veh/km/lane}.$$

The congested speed is given directly by the hydrodynamic relation,

$$V_{\text{cong}} = \frac{Q_{\text{cong}}}{\rho_{\text{cong}}} = 11.2\,\text{m/s} = 42.4\,\text{km/h}.$$

(d) With

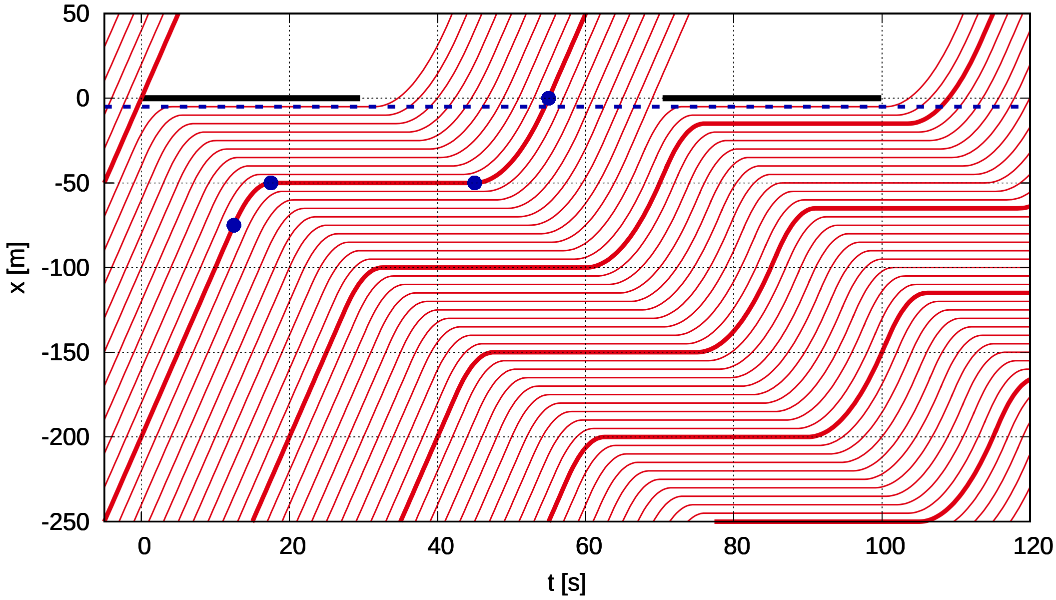
$$Q_1 = 0.5{*}4500/3600\,\text{veh/s/lane} = 0.625\,\text{veh/s/lane}, \quad \rho_1 = \frac{Q_1}{V_{01}} = 0.01875\,\text{veh/m/lane},$$

we obtain from the shockwave-formula

$$c_{12} = \frac{Q_1 - Q_{\text{cong}}}{\rho_1 - \rho_{\text{cong}}} = -3.52\,\text{m/s}.$$

Problem 2 (40 points)

Given are following trajectories:



(a) – Maximum density $\rho_{\text{max}} = 200\,\text{veh/km}$ (every tenth trajectory lies on a multiple of 50 m)

– Maximum outflow 0.5 veh/s (just count the five trajectories at $x = 50\,\text{m}$ after the thick one and read off the total passing time of 10 s).

(b) – Density 50 veh/km (10 lines between -250 m and -50 m)

– Flow 0.5 s (10 vehicles pass in an intervall of 20 s)

– Speed $V = Q/\rho = 10\,\text{s}$ (can a;lso be seen directly, e.g., by the gradient of the first thick trajectory entering the plotted area at $t = -5\,\text{s}$).

(c) The cycle time is $T_{\text{cyc}} = 70\,\text{s}$ and 17 vehicles pass in each cycle, hence, the cycle-average capacity is given by

$$C = 17/70\,\text{veh/s} = 0.243\,\text{veh/s},$$

which clearly is not enough for the demand $Q_{\text{in}} = 0.5\,\text{veh/s}$ (cf Problem (b)).

(d) Either directly from the graph:

$$w = \frac{-200\,\text{m}}{60\,\text{s}} = -3.33\,\text{m/s},$$

or by using the shockwave equation (optional!): We have (cf. Part (a) and realize that the outflow is equal to the demand using (b))

$$Q_{\text{out}} = 0.5\,\text{veh/s}, \quad Q_{\text{jam}} = 0, \quad \rho_{\text{out}} = 0.05\,\text{veh/m}, \quad \rho_{\text{jam}} = 0.2\,\text{veh/m}$$

Hence

$$w = \frac{Q_{\text{out}} - Q_{\text{jam}}}{\rho_{\text{out}} - \rho_{\text{jam}}} = -3.33\,\text{m/s}.$$

(e) The theoretical maximum number $n_{\text{theo}} = Q_{\text{out}} * T_{\text{green}} = 20$ while actually, $n = 17$ vheicles passed. Hence, the total delay with respect to uninterrupted flow is given by

$$T_{\text{delay}} = \frac{n_{\text{theo}} - n}{Q_{\text{out}}} = 6\,\text{s}.$$

(f) Estimates using a single stationary detector at $x = -5\,\text{m}$ ($V = 8\,\text{m/s}$ given):

$$Q = C = 0.243\,\text{veh/s}, \quad V = 8\,\text{m/s}, \quad \rho = \frac{Q}{V} = 0.0305\,\text{veh/m}.$$

(g) Edie's definition for $A = [-7.5\,\text{m}, -2.5\,\text{m}] \times [0\,\text{s}, 70\,\text{s}] = 350\,\text{sm}$ with $T_{\text{tot}} = 48\,\text{s}$ (given) and $X_{\text{tot}} = 85\,\text{m}$ (all 17 vehicles per cycle pass the 5 m section):

$$Q_{\text{Edie}} = \frac{X_{\text{tot}}}{A} = C = 0.243\,\text{veh/s}, \quad \rho_{\text{Edie}} = \frac{T_{\text{tot}}}{A} = 0.138\,\text{veh/m}, \quad V_{\text{Edie}} = 1.77\,\text{m/s}.$$

While the flow is consistent with the estimation from the stationary detector, the density is higher by a factor of more than 4 compared to the stationary detector estimate. Correspondingly, the speed is less than 1/4 of the detector estimate.

(h) The deceleration takes $T_b = 5\,\text{s}$ and the acceleration $T_a = 10\,\text{s}$ while $V_0 = Q_{\text{out}}/\rho_{\text{out}} = 10\,\text{m/s}$ (cf. (d)). Hence, we obtain

$$a = \frac{V_0}{T_a} = 1\,\text{m/s}^2, \quad b = \frac{V_0}{T_b} = 2\,\text{m/s}^2$$

The same results are obtained by observing that $\Delta x_a = 50\,\text{m}$, $\Delta x_b = 25\,\text{m}$, and using the acceleration/braking distance formulas $a = V_0^2/(2\Delta x_a)$ and $a = V_0^2/(2\Delta x_b)$, respectively.

Problem 3 (50 points)
Reference: IDM+ acceleration equations

$$\frac{\mathrm{d}v}{\mathrm{d}t} = f(s, v, v_l) = a \min\left[1 - \left(\frac{v}{v_0}\right)^4, \; 1 - \left(\frac{s^*(v, v_l)}{s}\right)^2\right]$$

with the dynamic desired gap

$$s^*(v, v_l) = s_0 + vT + \frac{v(v - v_l)}{2\sqrt{ab}}.$$

(a)
- **Input:** (bumper-to-bumper) gap s, follower's (subject vehicle) speed v, and leader's speed v_l.
- **Output:** The subject vehicle's acceleration $\frac{\mathrm{d}v}{\mathrm{d}t}$
- **Parameters:** that of the IDM, i.e., desired speed v_0, minimum gap s_0, desired time gap T, maximum acceleration a, and comfortable deceleration b
- **Workings:** The minimum function makes this model a two-regime model:
  - ∗ Free flow when the first minimum argument is relevant: No interactions, the acceleration depends only on the own speed v
  - ∗ Interacting flow: The second minimum counts and the acceleration depends on all three inputs.

(b) For the homogeneous steady state, we have $\frac{\mathrm{d}v}{\mathrm{d}t} = 0$ and $v_l = v$, so $s^*(v) = s_0 + vT$ and

$$0 = \min\left[1 - \left(\frac{v}{v_0}\right)^4, \; 1 - \left(\frac{s_0 + vT}{s}\right)^2\right].$$

- If the second condition is active (interacting regime), we require $1 - (v/v_0)^4 > 0$ and $1 - (s_0 + vT)/s = 0$, i.e., $v < v_0$ and $s_e(v) = s_0 + vT$ or $v_e(s) = (s - s_0)/T$
- If the first condition is relevant, we simply have $v = v_0$ and $s \geq s_0 + v_0 T$

In summary, we have the speed-gap relation

$$v_e(s) = \begin{cases} v_0 & s \geq s_0 + v_0 T \\ \frac{s - s_0}{T} & \text{otherwise.} \end{cases}$$

Converting $s = 1/\rho - l_{\text{eff}}$, we see that this is the speed-density relation corresponding to a tridiagonal fundamental diagram.

(c) The vehicle length is not an input of the equations of a car-following model such as the IDM+, so it cannot have an influence on the dynamics of a leader-follower pair or a string (platoon) of vehicles.[1]

(d) *Local stability* investigates if the steady-state gap of a follower following a leader with a fixed speed is stable, i.e., a *temporary* initial perturbation tends to zero over time (either oscillating or not).

String stability investigates whether *sustained oscillations* from a leader decreases when going from follower to follower.

[1] It has, however, an influence on the macroscopic properties such as the density $\rho = 1/(s + l_{\text{veh}})$ and the capacity, so changing the vehicle length can trigger a brekadown due to overload but it has no influence on stability (traffic waves or not).

(e) For free traffic, we have $f(s, v, v_l) = f_{\text{free}}(v) = a(1 - (v/v_0)^4$, so

$$f_s = 0, \quad f_v = \left.\frac{\mathrm{d} f_{\text{free}}}{\mathrm{d}v}\right|_{v=v_e} = -a\frac{4v^3}{v_0^4} = -\frac{4a}{v_0} < 0, \quad f_l = 0$$

and finally

- Local stability: $f_v < 0$:
- String stability: $v_e'(s) = 0 \le -f_v/2$:

(f) For $s \to s_0^+$, $v_e \to 0^+$, we have

$$f_s = \frac{2a}{s_0}, \quad f_v = -\frac{2aT}{s_0}, \quad f_l = 0$$

and the string stability condition reads

$$v_e' = \frac{1}{T} \le \frac{f_l - f_v}{2} = \frac{aT}{s_0} \quad \Rightarrow \quad a \ge \frac{s_0}{T^2}.$$

String stability near standstill is favoured by increasing a and T and decreasing s_0 (*this discussion is not needed.*)

(g) The general string stability condition in the interacting regime $s_e < s_0 + vT$ is given by (note that the $\sqrt{(a/b)}$ terms of f_v and f_l have the same magnitude but different signs)

$$v_e'(s) = \frac{1}{T} \le \sqrt{\frac{a}{b}}\frac{v_e(s)}{s} - \frac{aT}{s},$$

and with $v_e(s) = (s - s_0)/T$

$$v_e'(s) = \frac{1}{T} \le \frac{1}{Ts}\left(\sqrt{\frac{a}{b}}(s - s_0) + aT^2\right)$$

or

$$1 \le \sqrt{\frac{a}{b}}\left(1 - \frac{s_0}{s}\right) + \frac{aT^2}{s}.$$

With $a = b$ and $s > 0$, this simplifies to

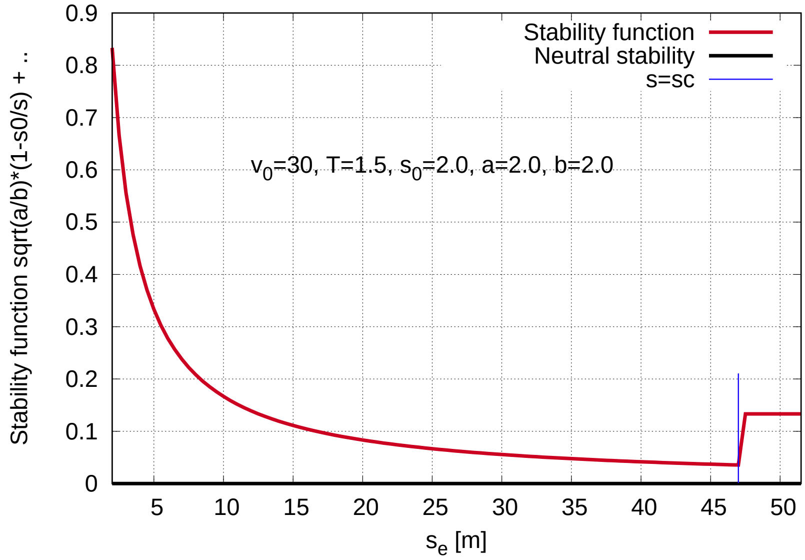
$$0 \le -s_0 + aT^2.$$

This means that, for $a = b$, the *complete* interacting range is stable if the condition $a \ge s_0/T^2$ is satisfied which is true for the given parameterisation. Since the free-flow regime is stable as well, this means that the IDM+ with this parameterisation is unconditionally string stable.

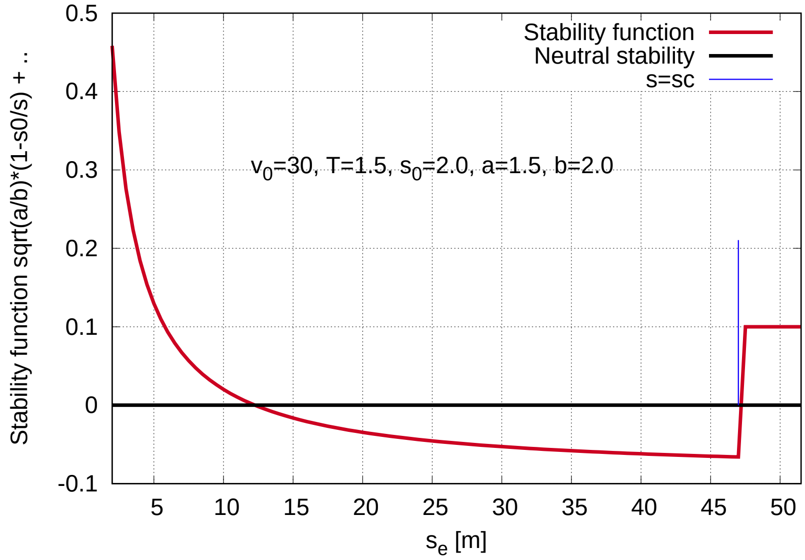
Side note (not part of the exam) For a more unstable parameterisation ($a = 1.5\,\mathrm{m/s^2}$ instead of $2\,\mathrm{m/s^2}$, there is still restabilisation near standstill ($a > s_0/T^2 = 0.889$) but string stability emerges for higher steady-state gaps as following plots of the string stability function

$$\S(s) = \frac{f_l - f_v}{2} - v_e'(s)$$

demonstrate.



Stability function
Neutral stability
s=sc
v0=30, T=1.5, s0=2.0, a=2.0, b=2.0
Stability function sqrt(a/b)*(1-s0/s) + ..
se [m]



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