Lecture 06: The Lighthill-Whitham-Richards (LWR) Model

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6.1 General: Motivation and Equations

- We have three main macroscopic quantities: density $\rho$, flow $Q$, and local speed $V$.
- There is always the static hydrodynamic relation between these quantities arising directly by the definitions of $\rho$, $Q$, and $V$:

$$Q = \rho V$$

- Furthermore, vehicle conservation implies the dynamic continuity equation, e.g., for a homogeneous road:

$$\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x} = \frac{\partial \rho}{\partial t} + \frac{\partial (\rho V)}{\partial x} = 0$$

So, two model-independent relations between the three quantities are always there. To make a macroscopic flow model that can be simulated, we need a third equation.
There are two basic possibilities to specify the missing third relation:

- **First-order** models or **Lighthill-Whitham-Richards (LWR) models** specify an additional static *traffic stream model/relation* between density and flow,
- **Second-order models** define a second dynamical equation for the speed

Inserting the fundamental diagram

\[ Q(x, t) = Q_e(\rho(x, t)) \]

as traffic stream relation into the continuity equation gives (for homogeneous roads) the class of first-order models known as LWR models:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \left( Q_e(\rho) \right) = 0
\]

\[
\text{LWR Model}
\]

? Show that the LWR model can also be written as

\[
\frac{\partial \rho}{\partial t} + (V_e + \rho V_e'(\rho)) \frac{\partial \rho}{\partial x} = 0.
\]
The earliest fundamental diagram of Greenshields

\[ Q(\rho) = V_0 \rho \left( 1 - \frac{\rho}{\rho_{\text{max}}} \right) \]
6.2 LWR Wave Velocity

Wave ansatz for solving the LWR equation
\[ \frac{\partial \rho}{\partial t} + \frac{\partial Q_e(\rho)}{\partial x} = 0: \]
\[ \rho(x, t) = \rho_0(x - ct) \]
\[ \frac{\partial \rho}{\partial t} = \rho_0'(x - ct) (-c) \]
\[ \frac{\partial Q_e(\rho)}{\partial x} = Q_e'(\rho) \rho_0'(x - ct) \]

This solves the LWR equation for all \( x \) and \( t \) iff

\[ -c + Q_e'(\rho) = 0 \]

or

\[ c = Q_e'(\rho) \quad \text{wave speed of small changes} \]

The wave speed is never larger than the vehicle speed: \( c = Q_e'(\rho) = V + \rho V_e'(\rho) \). Why? base your answer on plausibility criteria

Since there are only interactions front-back but not \textit{vice versa}
The density dependent wave speed \( c = Q_e' (\rho) \) means that the density can be imagined as layers (as in a 3d printer) independently gliding over each other until a shock is formed where the solution breaks down.
6.3. Formation of Shock Waves

Both the wave and the wave equation break down!
Derivation of the shock-wave propagation velocity

- Total vehicle number: \( n = \rho_1 x_{12} + \rho_2 (L - x_{12}) \)
- Rate of change as a function of the in- and outflows: \( \frac{dn}{dt} = Q_1 - Q_2 \)
- Rate of change as partial time derivative (watch out for the moving boundary with \( \frac{dc_{12}}{dt} = c_{12} \)):

\[
\frac{dn}{dt} = \frac{d}{dt} (\rho_1 x_{12} + \rho_2 (L - x_{12})) = (\rho_1 - \rho_2) \frac{dx_{12}}{dt} = (\rho_1 - \rho_2) c_{12} \Rightarrow \\
\]

\[
c_{12} = \frac{Q_2 - Q_1}{\rho_2 - \rho_1} = \frac{Q_e(\rho_2) - Q_e(\rho_1)}{\rho_2 - \rho_1} \quad \text{Shock-wave equation}
\]
6.3 Problems

Show that, in the case of a triangular fundamental diagram, the wave velocity is either equal to the vehicle speed or a constant negative value while the shock-wave propagation velocity can also takes on any value in between.

Triangular FD: $Q(\rho) = \min(Q_f(\rho), Q_c(\rho))$;
Free traffic: $Q_f(\rho) = V_0 \rho$, $c_f = Q'_f(\rho) = V_0 = \text{const.}$ (left slope);
Congested traffic: $Q_c(\rho) = 1/T(1 - \rho/\rho_{\text{max}})$, $w = Q'_c(\rho) = -1/(T\rho_{\text{max}}) = \text{const.}$ (right slope);
Shock velocity $c_{12}$: Slope of any line connecting the free with the congested side of the triangle, so $c \leq c_{12} \leq c_f$

What is the range of shock propagation velocities in the parabolic fundamental diagram of Greenshields?

Greenshields FD: $Q(\rho) = V_0 \rho (1 - \rho/\rho_{\text{max}})$, $Q'(\rho) = V_0 (1 - 2\rho/\rho_{\text{max}})$
⇒ both the wave and the shock velocities can take on values between $Q'(\rho_{\text{max}}) = -V_0$ and $Q'(0) = +V_0$
6.4 Triangular FD: Definition and Basic Properties

\[ Q_e(\rho) = \begin{cases} V_0 \rho & \text{if } \rho \leq \rho_c \\ \frac{1}{T} \left(1 - \rho l_{\text{eff}}\right) & \text{if } \rho_c < \rho \leq \rho_{\text{max}} \end{cases} \]

- Critical density: \( \rho_c = \frac{1}{V_0 T + l_{\text{eff}}} \)
- Maximum flow: \( Q_{\text{max}} = \frac{V_0}{V_0 T + l_{\text{eff}}} \)
- Maximum density: \( \rho_{\text{max}} = \frac{1}{l_{\text{eff}}} \)

Model parameters:
- Desired speed \( V_0 \)
- Effective vehicle length \( l_{\text{eff}} \) or maximum density \( \rho_{\text{max}} = \frac{1}{l_{\text{eff}}} \)
- Effective time gap \( T \) or wave speed \( w = -\frac{l_{\text{eff}}}{T} \)
## Typical parameters in different situations

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Highway</th>
<th>City Traffic</th>
<th>Pedestrian Single File</th>
</tr>
</thead>
<tbody>
<tr>
<td>Desired speed $V_0$</td>
<td>120 km/h</td>
<td>50 km/h</td>
<td>1.2 m/s</td>
</tr>
<tr>
<td>Time gap $T$</td>
<td>1.4 s</td>
<td>1.2 s</td>
<td>1.0 s</td>
</tr>
<tr>
<td>Max. density $\rho_{\max}$</td>
<td>120 veh/km</td>
<td>120 veh/km</td>
<td>1.5 peds/m</td>
</tr>
</tbody>
</table>

![Graphs showing flow vs. density for different situations](image-url)
Capacity as a function of the model parameters

- Highway traffic ($V_0=120$ km/h)
- City traffic ($V_0=50$ km/h)
- Living streets ($V_0=30$ km/h)

$Q_{\text{max}}$ [veh/h/lanes] vs. Time gap $T$ [s]
6.5 Properties of the Triangular FD

\[ Q(\rho) = \min \left( V_0 \rho, \frac{1 - \rho l_{\text{eff}}}{T} \right) \]

- **Analytic expression for the density at capacity:** \( \rho_c = \frac{1}{V_0 T + l_{\text{eff}}} \)
- **Analytic expression for the capacity:** \( Q_{\max} = V_0 \rho_c = \frac{V_0}{V_0 T + l_{\text{eff}}} \)
- **Fixed wave propagation velocities:** \( c(Q) = \begin{cases} V_0 & \text{free} \\ w = -\frac{l_{\text{eff}}}{T} & \text{congested} \end{cases} \)
- **Analytic inverse relations:**
  \[ \rho_{\text{free}}(Q) = \frac{Q}{V_0}, \quad \rho_{\text{cong}}(Q) = \frac{1 - QT}{l_{\text{eff}}} = \rho_{\max}(1 - QT) \]

- By means of the relations \( Q_{\max} = V_0/(V_0 T + l_{\text{eff}}) \) and \( w = -l_{\text{eff}}/T \), the unobservable quantities \( l_{\text{eff}} \) and \( T \) can be eliminated and the FD reformulated in terms of the observable parameters \( V_0, Q_{\max} \) and \( w \):

\[ Q_e(\rho) = \begin{cases} V_0 \rho & \text{if } \rho \leq \rho_c = \frac{Q_{\max}}{V_0} \\ Q_{\max} \left[ 1 - \frac{w}{V_0} \right] + w \rho & \text{if } \rho > \rho_c \end{cases} \]
In the triangular FD, waves in one regime (free or congested) remain unchanged.
Unique Properties of the Triangular FD (3)

The **upstream jam front** free → congested can be calculated by a time delayed ODE without solving the whole PDE using boundary conditions (e.g., from a detector) at both ends:

\[
    c_{12} = \frac{dx_{12}}{dt} = \frac{Q_1(t - \tau_f) - Q_2(t - \tau_c)}{\rho_1(t - \tau_f) - \rho_2(t - \tau_c)}
\]

- **\(Q_1(t)\)**: free traffic inflow from an upstream stationary detector
- **\(Q_2(t)\)**: congested outflow from a downstream stationary detector
- **\(\tau_f = (x_{12} - x_1)/V_0 > 0\)**: signal travel time from the upstream boundary to the front
- **\(\tau_c = (x_{12} - x_{\text{down}})/w > 0\)**: signal travel time from the downstream boundary to the front

The **downstream jam front** is either fixed at a bottleneck or moves upstream at velocity \(w\).
Application: State Detection and Short-Term Forecast: Upstream Jam Front

Both times $\tau_f$ and $\tau_c$ are positive $\Rightarrow$ real prediction based on vehicle conservation!
By calibrating the LWR parameters (essentially $w$ and $Q_{\text{max}}$ since $V_0$ has little influence), one obtains an *unbiased* estimate for $\rho_{\text{max}}$.
Infinite acceleration (softens for nontriangular FD)

The upstream front is always sharp irrespective of the FD and corresponds to an infinite deceleration
6.6 Bottlenecks – an overview

Types of bottlenecks:

▶ **Lane and flow-conservative bottlenecks**: no source terms, bottleneck effect only by spatial change of parameters

▶ **Lane closure bottlenecks**: bottleneck effect by source terms in the effective LWR while the LWR for the total quantities has no sources (flow-conservative)

▶ **Ramp bottlenecks**: bottleneck effect by source terms

▶ **Temporary bottlenecks** such as traffic lights.
Classic flow-conserving bottleneck

Ia  Ib  II  III
free upstream  jam  maximum flow  outflow
Classic flow-conserving bottleneck

The bottleneck with capacity $C_{bottl} = L Q_{\text{max}}$ has a different FD with a lower capacity than the other sections.

Definition of bottleneck: locally reduced capacity.

If congestion arises, the bottleneck emits information in both directions (why?).

$$\Rightarrow Q_{\text{Ib}} \text{ and } Q_{\text{III}} \text{ are equal to the bottleneck capacity } Q_{\text{II}}$$
Lane-closing bottleneck

La
free

Lb
jam

II
maximum flow
Lane-closing bottleneck

- Same FD per lane but different FD for total quantities
- Bottleneck capacity: \( C_{\text{bottl}} = L_2 Q_{\text{max}} \)
- Maximum-flow state in the whole downstream region, in contrast to the classic bottleneck
6.6 Bottlenecks – an overview

On-ramp bottleneck

Ia  | Ib  | II
free | jam | maximum flow
Traffic Flow Dynamics

6. The Lighthill-Whitham-Richards (LWR) Model

6.6 Bottlenecks – an overview

On-ramp bottleneck

- All road sections have the same FD but there is a source term in the continuity equation.
- Bottleneck capacity $LQ_{\text{max}} - Q_{\text{rmp}}$
- Continuous transition congested-maximum flow state in merging region

\[ \Delta C = Q_{\text{rmp}} \]
\[ \frac{L}{T} \left(1 - \frac{\rho_{\text{tot}} l_{\text{eff}}}{L}\right) \]

\( Q_{\text{tot}} \) vs \( \rho_{\text{tot}} \)

- 

\( L \rho_{\text{max}} \)
6.7 Traffic Lights: Temporary bottlenecks

- In the LWR approach, a traffic light is a temporary bottleneck with capacity zero.
- When the light becomes green, the stationary downstream jam front congested → free starts to move upstream at velocity $w$.
- Generally, downstream fronts are either “pinned” at the bottleneck or move upstream at velocity $w$.
- There is a single situation where a downstream front may move downstream. Which?
Calculation of the total waiting time in one red phase

- \( \rho_{\text{max}} \) times the vertical extension of the red triangle (e.g., from E to D) gives the actual number of stopped vehicles per lane.

- The area of the triangle gives the total waiting time per lane.

- This area can be easily calculated by adding the areas of the two rectangular triangles ODE and DEF'.
Draw a triangular FD with the traffic states ① to ④ and also denote graphically the different shock-wave propagation velocities.
6.8 Examples: 1.accident

- Single-lane obstruction between 15:00 and 15:30 pm
- Constant inflow 3024 veh/h
- Triangular FD parameters $l_{\text{eff}} = 8 \text{ m}$, $T = 1.5 \text{ s}$, and $V_0 = 28 \text{ m/s}$

1. Does the road capacity prior to the accident satisfy the demand?
   - Maximum flow per lane: $Q_{\text{max}} = \frac{V_0}{V_0T+l_{\text{eff}}} = 2016 \text{ veh/h}$.
   - Capacity before the accident: $C = 2Q_{\text{max}} = 4032 \text{ veh/h}$.
   - This exceeds the traffic demand 3024 veh/h, so no jam.
2. Show that the accident leads to a traffic breakdown. Calculate the total and effective flows in all sections.

- Bottleneck capacity \( C_{bottl} = Q_{max} = 2016 \text{ veh/h} \) is smaller than inflow \( Q_{in} \) \( \Rightarrow \) traffic breakdown.

- Upstream free flow controlled by inflow, \( Q^{tot}_1 = 3024 \text{ veh/h} \), and congested flow as well as the flow in all following segments by the bottleneck: \( Q^{tot}_2 = Q^{tot}_3 = C_{bottl} = 2016 \text{ veh/h} \)

- Per lane, the effective flows are \( Q_1 = 1512 \text{ veh/h} \), \( Q_2 = Q_3 = 1008 \text{ veh/h} \)
3. Calculate the propagation velocity of the upstream jam front

- Besides the flow, we need the densities. They are given by the suitable branch of the inverse FD:
  \[ \rho_1 = \frac{Q_1}{V_0} = 15 \text{ veh/km}, \quad \rho_2 = \rho_{\text{cong}}(Q_2) = \frac{1 - Q_2 T}{l_{\text{eff}}} = 72.5 \text{ ve/km} \]

- Propagation velocity of upstream jam front:
  \[ c_{\text{up}} = c_{12} = \frac{Q_2 - Q_1}{\rho_2 - \rho_1} = -8.77 \text{ km/h} \]
4. Calculate the velocity of the moving downstream front once the obstruction has been removed and the time for complete dissolution of the jam.

Once the obstruction has been removed, the maximum flow state always arising at the downstream end of jams is over both lanes, so the new flows downstream of the congestion are

\[ Q_4^{\text{tot}} = C \quad \text{and} \quad Q_4 = C/2 = Q_{\text{max}} = 2016 \text{ veh/h} \]

and, from the free branch,

\[ \rho_4 = Q_4/V_0 = 20 \text{ vehicles/h}. \]

Thus,

\[ c_{\text{down}} = c = c_{23} = \frac{Q_4 - Q_2}{\rho_4 - \rho_2} = -19.2 \text{ km/h} \]
5. Visualize the spatiotemporal dynamics of the jam by drawing its boundaries in a space-time diagram.

6. How much time does a vehicle need to traverse the 10 km long road section if it enters at 15:30 h?
Example 2: Uphill Grade and Lane Drop, Q1

Triangular FD (per lane) as follows:

\[ V_0 = \begin{cases} 
60 \text{ km/h} & \text{Section III} \\
120 \text{ km/h} & \text{other sections} 
\end{cases} \]
\[ T = \begin{cases} 
1.9 \text{ s} & \text{III} \\
1.5 \text{ s} & \text{others} 
\end{cases} \]
\[ l_{\text{eff}} = 10 \text{ m} \]

1. Calculate the local road capacity and identify possible bottlenecks

! capacity is number of lanes times \( Q_{\text{max}} \) where \( Q_{\text{max}} \) is equal for the Sections I, II, and IV; bottlenecks are local capacity drops, i.e., beginning of the Sections II and III → tutorial to this lecture
Example 2: Uphill Grade and Lane Drop, Q2

Triangular FD (per lane) as follows:

\[ V_0 = \begin{cases} 
60 \text{ km/h} & \text{Section III} \\
120 \text{ km/h} & \text{other sections}
\end{cases}, \quad T = \begin{cases} 
1.9 \text{ s} & \text{III} \\
1.5 \text{ s} & \text{others}
\end{cases}, \quad l_{\text{eff}} = 10 \text{ m}
\]

2. At 4:00 pm, the total traffic demand at \( x = 0 \) increases abruptly from 2000 veh/h to 3600 veh/h. Does this cause a breakdown? If so, at which time and where?

! Answer: Check where, when going from upstream to downstream (why?) the demand exceeds the capacity for the first time. The relevant bottleneck is said to be activated.
Example 2: Uphill Grade and Lane Drop, Q3

Triangular FD (per lane) as follows:

\[ V_0 = \begin{cases} 
60 \text{ km/h} & \text{Section III} \\
120 \text{ km/h} & \text{other sections}
\end{cases}, \quad T = \begin{cases} 
1.9 \text{ s} & \text{III} \\
1.5 \text{ s} & \text{others}
\end{cases}, \quad l_{\text{eff}} = 10 \text{ m} \]

3. Calculate the dynamics of the developing congestion if the inflow remains constant

! The downstream front is pinned at the activated bottleneck. The upstream front is determined by the shockwave formula. You need to distinguish the propagation in the Regions II and I! → tutorial to this lecture
6.9 Numerics of the LWR

I: Mathematical background

Mathematically, the LWR model equation $\frac{\partial \rho}{\partial t} + \frac{\partial Q_e(\rho)}{\partial x}$ is a hyperbolic partial differential equation (PDE) for $\rho(x, t)$ in the special form of a conservation law. This PDE can be solved (the problem is well posed; the solution exists and is unique as the math people say) provided

- The initial condition $\rho(x, 0)$ at time $t = 0$ is completely known for all $x \in [0, L_{\text{road}}]$ along the road
- In case of free traffic $\rho(0, t) < \rho_c$, the upstream boundary condition (BC) $\rho(0, t) = \rho_{\text{free}}(Q_{\text{up}}(t))$ is given by the traffic demand $Q_{\text{up}}$ per lane
- In case of a downstream congestion, the downstream BC $\rho(L_{\text{road}}, t) = \rho_{\text{cong}}(Q_{\text{down}}(t))$ is determined by the maximum flow this congestion can take.

When solving a conservation law, it is crucial to take into account the direction of information flow.,

Depending on the situation, 0, 1, or 2 BC apply.
Traffic Flow Dynamics

6. The Lighthill-Whitham-Richards (LWR) Model

6.9 Numerics of the LWR

Cell length $\Delta x$, update time interval $\Delta t$, approximated density $\rho_{kj} = \rho(k\Delta x, j\Delta t)$

take into account signal propagation directions
General LWR models (ctnd): discretisation in space

- Approximate the spatial derivative $\frac{\partial Q}{\partial x}$ by first-order finite differences taking account the signal propagation (essentially Godunov’s method)

- If information propagates downstream (free traffic), use downwind finite differences (“with the wind, with the stream”) for $\frac{\partial}{\partial x}$ to “catch” this information:

  $$\frac{\partial Q(k \Delta x, j \Delta t)}{\partial x} \approx \frac{Q_{k,j} - Q_{k-1,j}}{\Delta x}$$

- If information propagates upstream (congested traffic), use upwind finite differences (“against the wind”):

  $$\frac{\partial Q(k \Delta x, j \Delta t)}{\partial x} \approx \frac{Q_{k+1,j} - Q_{k,j}}{\Delta x}$$
General LWR models (ctnd): discretisation in time

- **Explicit** numerical scheme: only past and present needed
- **First-order** scheme: errors for integrating a certain time interval decrease linearly with $\Delta t$ and $\Delta x$ if both tend to zero
- The **Euler method** is the simplest of such schemes: $f(t + \Delta t) \approx f(t) + f'(t)\Delta t$ for any $f(t)$

\[
\rho_{k,j+1} = \rho_{k,j} - \frac{\Delta t}{\Delta x} \left\{ \begin{array}{c}
(Q_{k-1,j} - Q_{k,j}) \\
(Q_{k+1,j} - Q_{k,j})
\end{array} \right\}
\]

\[
Q_{k,j+1} = Q_e(\rho_{k,j+1}) \quad \text{free}
\]

\[
Q_{k,j+1} = Q_e(\rho_{k,j+1}) \quad \text{congested}
\]

**Courant-Friedrichs-Lévy (CFL)** stability criterion:

\[
\Delta t \leq |c|_{\text{max}} \Delta x = V_0 \Delta x
\]
Numerics III: Supply-Demand-Method for Triangular FDs

General scheme also applies to triangular FDs. However, near capacity, it becomes tricky to determine whether downwind or upwind finite differences to apply (why?). The supply-demand method gives a specialized simplified procedure for triangular FDs:

1. Define the supply (maximum flow the downstream cell can receive) and demand (maximum flow the upstream cell can deliver) functions with capacity $C_k = L_k Q_{\text{max}}$:

   $$S(k) = \min(C_k, L_k Q_{\text{cong}}(\rho_k)),$$
   $$D(k) = \min(C_k, L_k Q_{\text{free}}(\rho_k))$$
### Supply-Demand-Method for Triangular FDs (ctnd)

<table>
<thead>
<tr>
<th>Capacity</th>
<th>( C_{k-1} )</th>
<th>( C_k )</th>
<th>( C_{k+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q_{tot}^{up} )</td>
<td>( Q_k )</td>
<td>( Q_{tot}^{down} )</td>
<td>( Q_{tot}^{up} )</td>
</tr>
</tbody>
</table>

2. As in any trading, the actual flow (amount of delivered products) is given by the minimum of supply and demand. For the two boundaries of cell \( k \):

\[
Q_{k}^{tot,up} = Q_{k-1}^{tot,down} = \min (S_k, D_{k-1}) ,
\]

\[
Q_{k}^{tot,down} = Q_{k+1}^{tot,up} = \min (S_{k+1}, D_k) .
\]

3. Explicit first-order as in the general case:

\[
\rho_k(t + \Delta t) = \rho_k(t) + \frac{1}{L_k \Delta x_k} \left( Q_{k}^{tot,up} - Q_{k}^{tot,down} \right) \Delta t ,
\]

\[
Q_k(t + \Delta t) = Q_e (\rho_k(t + \Delta t)) .
\]
Cell-transmission model for networks

\[ S_3 = \min(C_3, L_3 Q_{\text{cong}}(\rho_3)), \]
\[ D_3 = \min(C_3, L_3 Q_{\text{free}}(\rho_3)), \]
\[ D_{12} = \min(C_1, L_1 Q_{\text{free}}(\rho_1) + \min(C_2, L_2 Q_{\text{free}}(\rho_2)), \]
\[ Q_{3}^{\text{tot,up}} = \min(S_3, D_{12}), \]
\[ Q_{3}^{\text{tot,down}} = \min(S_4, D_3), \]
\[ \rho_{3}^{\text{tot}}(t + \Delta t) = \rho_{3}^{\text{tot}} + \frac{1}{\Delta x_3} \left( Q_{3}^{\text{tot,up}} - Q_{3}^{\text{tot,down}} \right) \Delta t \]

In case of congestion, only the sum \( Q_{3}^{\text{tot,up}} \) is defined and the supply distributed to the demands \( D_1 \) and \( D_2 \) according to traffic regulations/priority rules.
Cell transmission model for other bottlenecks

- **Bottlenecks in the stricter sense** and also changes in the number of lanes? are automatically included in the supply-demand model for straight roads.

- **Merge bottlenecks**? are just two-in-one nodes where the priority (in contrast to regulations) is given to the ramp.

- **Diverges**? such as cell 1 → cells 2, 3 require the diverging fraction $\lambda_3$ as additional input. Congestion arises if $\lambda_3 D_1 > S_3$ or $(1 - \lambda_3) D_1 > S_2$. If $\lambda_3 D_1 > S_3$, we have

$$Q_{1 \text{tot,down}} = \frac{S_3}{\lambda_3}$$

leading to a **spill-back bottleneck**.

1. Make this formula plausible! On average, a flow $S_3/\lambda_3$ can pass the diverge such that Link 3 is at its supply limit $S_3$. The excess link-3 drivers wait until they can enter thereby obstructing also the upstream link-2 drivers.

2. Discuss lane changes as sources of off-ramp bottlenecks. Lane changes disturb the flow reducing the maximum flow. This cannot be modelled by LWR models unless special provisions are taken.