Traffic Flow Dynamics and Simulation

Summer semester, Solutions to Work Sheet 12, page 1

Solution to Problem 12.1: Dynamic Navigation

(a) Travel times. In an empty network, the maximum speed V_0 (which here is the same on both routes) can be driven, so

$$T_{01} = \frac{L_1}{V_0} = 750\,\text{s}, \quad T_{02} = \frac{L_2}{V_0} = 800\,\text{s}.$$

(b) (i) User equilibrium I: Qualitative considerations. Once an inflow $Q_\text{in} = 5,400\,\text{veh/h}$ arrives, all drivers will first choose R1 since $T_{10} < T_{20}$. However, since $Q_\text{in} > C_1^\text{B}$, a traffic jam will form behind the bottleneck. New incoming drivers with a perfect knowledge of the instantaneous travel times on both routes will still select R1 until the travel time T_i has increased near T_{20}. Then, some drivers will select R2 such that the length of the jam on R1 remains constant at $T_1 \approx T_{20}$. However, this will only happen in the presence of noise, $\sigma_T > 0$. Otherwise, we have a *bang-bang control* which, in the presence of delays, will always lead to oscillations. Since there is no hysteresis or capacity drop in LWR models, the flow on R1 in the user equilibrium (UE) and the corresponding flow on R2 are given by

$$Q_1^\text{tot,UE} = C_1^\text{B} = 4,860\,\text{veh/h}, \quad Q_2^\text{tot,UE} = Q_\text{in} - Q_1^\text{tot,UE} = 540\,\text{veh/h}.$$

The resulting flow $Q_2^\text{tot,UE}$ is well below the bottleneck capacity C_2^B, so no jam will form on R2.

(ii) Length of the congestion on R1 in user equilibrium. The length x of the congestion on R1 in user equilibrium is given by the condition of equal travel times on both routes:

$$T_1 = \frac{x}{V_\text{cong}} + \frac{L_1 - x}{V_0} \stackrel{!}{=} T_{20},$$

or

$$x = \frac{T_{20} - \frac{L_1}{V_0}}{\frac{1}{V_\text{cong}} - \frac{1}{V_0}}.$$

In order to obtain the speed V_cong in the congestion, we observe that the congested flow per lane on the three-lane road is given by $Q_\text{cong} = C_1^\text{B}/3$. Since the wave speed w is the slope of the congested branch of the triangular fundamental diagram, and $Q_\text{cong} = 0$ for $\rho = \rho_\text{max}$, the associated density is determined by $Q_\text{cong} = w(\rho - \rho_\text{max})$ or

$$\rho_\text{cong} = \rho_\text{max} + \frac{Q_\text{cong}}{w} = \rho_\text{max} + \frac{C_1^\text{B}}{3w} = 60\,\text{veh/km},$$

whence

$$V_\text{cong} = \frac{Q_\text{cong}}{\rho_\text{cong}} = 7.5\,\text{m/s}, \quad x = 600\,\text{m}.$$

(c) System optimum (SO). Since $T_{10} < T_{20}$, the system optimum implies that as many vehicles as possible take R1 without producing a jam. This means $Q_1^{\text{tot,SO}} = C_1^{\text{B}}$ as in the user equilibrium. However, vehicles are sent on R2 *before* a jam can form, so that $T_2 > T_1$ which means, the SO is no equilibrium configuration, whether stable or unstable. In summary, we have

$$Q_1^{\text{tot,SO}} = Q_1^{\text{tot,UE}} = 4,860\,\text{veh/h}, \quad Q_2^{\text{tot,SO}} = Q_2^{\text{tot,UE}} = 540\,\text{veh/h},$$

and

$$\begin{aligned} T_1 &= T_{10} = 750\,\text{s}, \quad T_2 = T_{20} = 800\,\text{s}, \\ \bar{T} &= \frac{Q_1^{\text{tot,SO}}}{Q_{\text{in}}} T_1 + \frac{Q_2^{\text{tot,SO}}}{Q_{\text{in}}} T_2 = 0.9T_1 + 0.1T_2 = 755\,\text{s}. \end{aligned}$$

(d) *Necessary conditions for a UE:* As calculated previously, both the UE and SO require that a percentage

$$P_2^{\text{SO}} = \frac{Q_{\text{in}} - C_1^{\text{B}}}{Q_{\text{in}}} = 10\%$$

diverts to the deviation. If the uncertainty σ_T is sufficiently small, $\sigma_T \ll T_{02} - T_{01}$, all equipped drivers will divert as soon as the jam on Route 1 reaches its UE length but not earlier thus stabilizing the UE. Obviously, this is only possible if there are enough equipped vehicles, $\alpha > P_2^{\text{SO}}$. If, however, α is significantly greater than P_2^{SO}, oscillation instabilities will form. If σ_T is no longer $\ll T_{02} - T_{01}$, oscillations are suppressed but some drivers will already use Route 2 before the UE length of the jam is reached leading to a situation between the UE and SO.

Necessary conditions for a SO: In order that a SO is sustained, a fraction P_2^{SO} of drivers must use Route 2 right at the beginning, i.e., use a route which is longer by the time difference $T_{02} - T_{01}$. In order to do this, the uncertainty must be at a certain level given by

$$P_2 = \frac{\alpha}{e^{V_{01} - V_{02}} + 1} = \frac{\alpha}{e^{\frac{T_{02} - T_{01}}{\sigma_T}} + 1}$$

which can be solved for α resulting in the SO condition of the problem statement,

$$\alpha = P_2^{\text{SO}} \left(e^{\frac{T_{02} - T_{01}}{\sigma_T}} + 1 \right). \tag{1}$$

Additionally, we must require $P_2 < 50\%$ because, otherwise, Condition (1) can never be met.

(e) Necessary conditions for no jam on Route 2. Congestions on both routes are avoided in the steady state if $P_s^{\text{SO}} \le P_2 \le P_2^{\max}$ where $P_2^{\max}$ is calculated from the maximum flow $Q_{\text{in}} P_2^{\max} = C_2^{\text{B}}$ that Route 2 can accommodate without becoming congested, so

$$P_2^{\max} = \frac{C_2^{\text{B}}}{Q_{\text{in}}} = 20\%.$$

In terms of α, this means

$$\alpha \le P_2^{\max} \left(e^{\frac{T_{02} - T_{01}}{\sigma_T}} + 1 \right).$$