Traffic Flow Dynamics and Simulation

Summer semester, Solutions to Work Sheet 11, page 1

Solution to Problem 11.1: Pilgrimage in Mekka

(a) *Density:* Pedestrians are moving in a 2d area, so only a definition of the density as # pedestrians per m/s^2 makes sense:

$$[\rho] = 1\,\mathrm{m}^{-2}$$

(of course, in a ingle-file environemnt, we still have #per per meter) *Flow:* For a cross section (width) of the pedestrian pasageway of more than about $W = 1.5\,\mathrm{m}$, the flow [# pedestrians per second] at a given density increases proportional to the width W.[1]. So, in this case, the flow Q [#ped/s] scales as

$$Q = Jw$$

where the proportionality factor denotes the additional flow per meter of cross section. Hence, it is appropriately named **flow density**

$$[J] = 1\,\mathrm{ms}$$

J denotes the number of pedestrians per second and per meter of cross section.[2]

(b) Typically, when observing pedestrian flow, one records a video from a fixed position (or a drone) and extracts trajectories $(x_i(t), y_i(t))$ for all pedestrians out of it. Because of this full microscopic information, systematic errors can be avoided by appropriate analysis/interpretation, e.g., as described in part (b).

Notice that for irregularly moving pedestrians (as on Christmas markets), only density but no flow or flow density makes sense

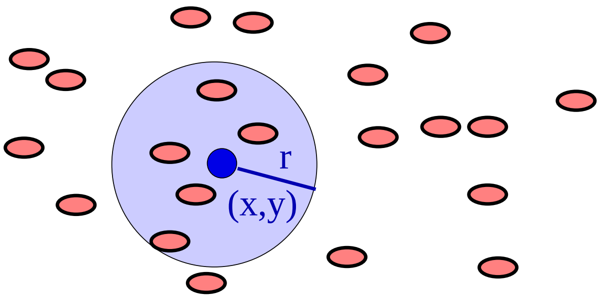
(c) **Extracting the macroscopic quantities from tajectory data**

- **Density** $\rho(x, y, t)$ Use a video still at time t and count all pedestrians that are inside a circle of radius r centered at (x, y):

$$\rho(x, y, t) = \frac{\#\text{pedetrians in circle}}{\pi r^2} = \frac{\sum_i 1}{\pi r^2}$$

[1] This is valid for unidirectional use; for bidirectional use, the minimum width is larger; for pedestrians moving in all directions, the concept *flow* is not well defined

[2] In lane-based vehicular traffic, this proportionality corresponds to the proportionality of the capacity with the number of lanes. In fact, one could define a flow density $J = Q/W_{\text{lane}}$ by dividing the flow by the lane width W_{lane}.



* $r \approx 3\,\text{m}$ should be microscopically large (contains several pedestrians) and macroscopically small (smaller than the scale of the spatial structures to be resolved)
* You could also use more refined kernels, e.g., cone-like or 2d Gaussian
* Alternatively, you could define

$$\rho(x, y, t) = 1/A_{\text{voronoi}}$$

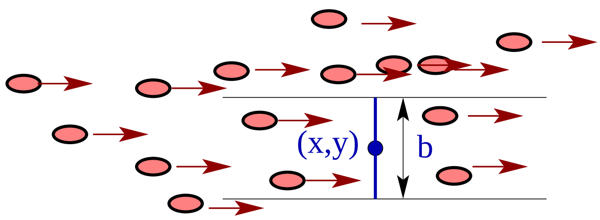
where A_{voronoi} is the area of the Voronoi cell containing the point (x, y)

– **Local space-mean speed/velocity** $V(x, y, t)$: just take the arithmetic average of the speed/velocity of the pedestrians inside the circle or the speed of the pedestrian in its Voronoi cell:

$$V(x, y, t) = \frac{\sum_i v_i}{\sum_i 1}$$

– **Flow density** $J(x, y, t)$: define a line of length $L < W$ perpendicular to the flow and centered at (x, y) and determine the #pedestrians crossing this line in the time interval $[t - \tau/2, t + \tau/2]$:

$$J(x, y, t) = \frac{\text{\#pedestrians crossing}}{L\tau}$$



L and τ should be microscopically large and macroscopically small, e.g., $L = 2\,\text{m}$, $\tau = 5\,\text{s}$.

(d) **(i) Approximate parameters of the triangular fundamental diagram**

$$J(\rho) = \min\left(\rho V_0, -(\rho_{\max} - \rho)w\right)$$

European event (blue flow-density data):

– Maximum density by the intersection of the extrapolated flow-density points with the x axis:

$$\rho_{\text{max}} = 5.5\,\text{ped/m}^2$$

(no bias expected since the true spatial density can be estimated from the data)

– Desired speed from the gradient of the flow-density data for very low densities:

$$V_0 = 1\,\text{m/s}$$

– Wave velocity w by the gradient of the points on the congested side:

$$w = -2\,\text{ms})^{-1}/\rho_{\text{max}} = -0.35\,\text{m/s}$$

(Notice that the time gap T does not make sense in the 2d context. It would have the unit sm (second time meter) and the value (intersection of the congested branch with the y axis) of about $T = 2\,\text{sm}$. Hence, it is better to use the wave velocity w as the primary parameter.)

Hajj pilgrimage: A triangular FD does not make sense

(ii) Approximate parameters for the parabolic Greenshields fundamental diagram

$$J(\rho) = V_0\rho\left(1 - \frac{\rho}{\rho_{\text{max}}}\right)$$

European event (blue flow-density data):

– Maximum density: A bit lower than for the triangular case, $\rho_{\text{max}} \approx 5\,\text{ped/m}^2$

– Desired speed: Such that the observed specific capacity $J_{\text{max}} = 1.5\,\text{ped/(sm)}$ is the same as the theoretical one,

$$J_{\text{max}} = \frac{V_0\rho_{\text{max}}}{4} \quad \Rightarrow \quad V_0 = \frac{4J_{\text{max}}}{\rho_{\text{max}}} = 1.5\,\text{m/s}$$

Hajj pilgrimage:

$$\begin{aligned} \rho_{\text{max}} &= 10\,\text{m/s}^2, \\ V_0 &= \frac{4J_{\text{max}}}{\rho_{\text{max}}} = \frac{8\,/(\text{ms})}{10\,\text{m/s}^2} = 0.8\,\text{m/s} \end{aligned}$$

Notice that there is a discrepancy between this estimation and the gradient of the points for density to zero (about 1 m/s) which arises from the fact that neither the triangular not the Greenshields FD fit the data.[3]

[3] These data look like there was a measuring artifact/a wrong estimation method for the high values or the density, or that the pedestrian flow entered in a new phase.

(e) This can be best explained for the free-flow part, where the isotropic condition $\rho = \rho_{1d}^2$ gives the free-flow relation

$$Q = JL = \rho V_0 L = \rho_{1d}^2 V_0 L,$$

increasing *quadratically* with the 1d density. This strange result becomes plausible when realizing that, at very low densities, doubling the 1d density means a double flow within each single file *and* also the double number of single files.

In effect, a single file just needs the lateral space

$$W_{\text{ped}} = \frac{1}{\sqrt{\rho_{\max}}},$$

so, for single-file dynamics, we effectively have

$$\rho = \rho_{1d} \, \frac{1}{W_{\text{ped}}}, \quad \rho_{\max} = \rho_{1d,\max} \, \frac{1}{W_{\text{ped}}}$$

and for the triangular FD the flow

$$Q_{\text{single}} = W_{\text{ped}} J = W_{\text{ped}} \min(\rho V_0, -(\rho_{\max} - \rho) w) = \min(V_0 \rho_{1d}, -(\rho_{1d,\max} - \rho) w)$$

with $\rho_{1d,\max} = 1/W_{\text{ped}}$ and the value $\rho_{1d,\max} = 2.3\,\text{ped/m}$ for the European event.

This results in an unchanged wave velocity w and the effective desired time gap for congested single files:

$$T = -\frac{1}{w \rho_{\max,1d}} \approx 1,2\,\text{s}$$

Remarkably, the implicit time gap for pedestrians is of the same order as that for cars in city or freeway traffic

Solution to Problem 11.2: Social Force Model

(a)
- B: range of the interaction: If one is sufficiently far away from the target pedestrian (distance $|\vec{x}| \gg |\delta\vec{d}|$), the minor semi-axis b essentially becomes equal to the distance because, then, the ellipse with focal points a distance $\delta\vec{d}$ from each other and containing the point $\vec{x}$ converges to a circle of radius $|\vec{x}| = b$. Therefore, the potential and the force decays by a factor of $1/e$ when the distance is increased by B. Experience tells us that a decay distance $B = 1\,\mathrm{m}$ is plausible.
- A: strength of the interaction. As a minimal condition, the interaction should prevent passing-throughs of pedestrians (they are no ghosts!) requiring $v^2/2 < \Phi^{\mathrm{int}}(0) = AB$ (cf problem Part (c)). Here, we have $AB = 2\,\mathrm{m}^2/\mathrm{s}^2$ preventing such collisions for speeds below $v_c = \sqrt{2AB} = 2\,\mathrm{m/s} = 7.2\,\mathrm{km/h}$. This should also be enough when considering that the minimum distance is about $0.5\,\mathrm{m}$ and also including the additional driving force of the desired speed (provided τ is not too small). Another plausibility test is the maximum deceleration given by A as well. This also makes a value $A = 2\,\mathrm{m/s}^2$ plausible.
- Δt: anticipation time horizon for the trajectory planning. As in car driving, it should be comparable to the reaction time and the time gap when following each other: OK
- λ: anisotropy: The repulsive force is weakened by a factor λ when the target is not straight ahead but behind and partially weakened for other directions. Should be $\ll 1$, even zero is plausible: OK.
- τ: free-flow adaptation time. Determines, together with $\vec{v}_0$, the free-flow dynamics. Essentially measures the time a pedestrian needs to reach his/her desired walking speed from standstill which is plausible. Also gives the maximum free acceleration $v_0/\tau = 0.75\,\mathrm{m/s}^2$ which is plausible as well. Finally, the maximum free-flow acceleration is lower than the maximum interaction acceleration of the order of $A = 2\,\mathrm{m/s}^2$

(b) Calculating the interaction potential essentially entails calculating the semi-axis $b(\vec{x})$. For straight-ahead approaches to a standing obstacle as in Situation I assuming $\vec{x} = (-r, 0)$, $\vec{v} = (v, 0)$, $r > 0$, and $r > v\Delta t$, the equation for b simplifies as follows:

$$
\begin{aligned}
b(r|v) &= \frac{1}{2}\sqrt{\left(|\vec{x}| + |\vec{x} + \Delta\vec{d}|\right)^2 - |\Delta\vec{d}|^2} \\
&= \frac{1}{2}\sqrt{(r + |\vec{x} + \vec{v}\Delta t|)^2 - (v\Delta t)^2} \\
&= \frac{1}{2}\sqrt{(r + (r - v\Delta t))^2 - (v\Delta t)^2} \\
&= \frac{1}{2}\sqrt{(2r - v\Delta t)^2 - (v\Delta t)^2} \\
&= \frac{1}{2}\sqrt{4r^2 - 4rv\Delta t} \qquad = \sqrt{r^2 - rv\Delta t}
\end{aligned}
$$

Inserting the values of Situation 1 gives with $\Delta t = 1\,\mathrm{s}$ the semi-minor axis $b = 2.12\,\mathrm{m}$

and

$$\Phi^{\text{int}} = AB\ \exp(-b/B) = 0.240\,\text{m}^2/\text{s}^2$$

(c) For a pedestrian moving along the x axis at position $x = -r < 0$ and a standing target pedestrian at the origin, we have $w(\phi) = 1$, so

$$\begin{aligned}\frac{\mathrm{d}v}{\mathrm{d}t} &= \frac{v_0 - v}{\tau} - \frac{\mathrm{d}\Phi^{\text{int}}}{\mathrm{d}x} \\ &= \frac{v_0 - v}{\tau} + Ae^{-b/B}\frac{\mathrm{d}b}{\mathrm{d}x} \\ &= \frac{v_0 - v}{\tau} - Ae^{-b/B}\frac{\mathrm{d}b}{\mathrm{d}r} \\ &= \frac{v_0 - v}{\tau} - Ae^{-b/B}\ \frac{r - v\Delta t/2}{b}\end{aligned}$$

For Situation I, we have $(v_0 - v)/\tau = 0$ (the pedestrian moves at its desired velocity), $b = 2.12\,\text{m}$ from above, $r = 3\,\text{m}$ and $v\Delta t/2 = 0.75\,\text{m}$, so

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -0.254\,\text{m/s}^2$$

(d) (i) If the potential depends only on $\vec{x}$ and not on the speed (of the subject pedestrian) or explicitly on time (if the target pedestrian moves), the SFM acceleration for $\tau \to \infty$ and without the directionality $w(\phi)$ (e.g., for frontal approaches) reads

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = -\nabla\Phi^{\text{int}}(\vec{x})$$

or for Situation I:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{\mathrm{d}\Phi^{\text{int}}(x)}{\mathrm{d}x}$$

Multiplying both sides with $v = \frac{\mathrm{d}x}{\mathrm{d}t}$ gives

$$\begin{aligned}v\frac{\mathrm{d}v}{\mathrm{d}t} &= -\frac{\mathrm{d}\Phi^{\text{int}}(x)}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v^2}{2}\right) &= -\frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi^{\text{int}}(x(t))\right) \qquad \int \mathrm{d}t \\ \frac{v^2}{2} &= -\Phi^{\text{int}}(x) + \text{const.}\end{aligned}$$

or

$$E = \frac{v^2}{2} + \Phi^{\text{int}}(x) = \text{const.}$$

(ii) We calculate the constant energy from the initial conditions:

$$E = \Phi^{\text{int}} + \frac{1}{2}v^2 = 1.365\,\text{m}^2/\text{s}^2 < AB = 2\,\text{m}^2/\text{s}^2$$

This means, Pedestrian 1 stops before reaching the center of Pedestrian 2 [4]

[4] with anticipation, there is even a larger marging but it is not possible to calculate it analytically.

(iii) Without anticipation, we have $\Delta\vec{d} = \vec{0}$ and the equation for the semi-minor axis becomes (for a target pedestrian at the origin)

$$b = |\vec{x}| = r$$

For the stopped pedestrian, $v = 0$ but its energy is still the same as in the beginning, so we have

$$\begin{aligned} E &= \Phi^{\text{int}}(x) = ABe^{-x/B} \\ \ln\left(\frac{E}{AB}\right) &= -\frac{x}{B}, \end{aligned}$$

giving the stopping distance

$$x = x_s = -B\ln\left(\frac{E}{AB}\right) = 0.382\,\text{m}.$$

(e) The target pedestrian is standing at the origin (like an unpenetrable obstacle). Since the subject pedestrian already has a negative y-component, he/she will surely swerve further to the right to avoid the target, i.e., the y-component of the acceleration should be negative.

Since the pedestrian walks presently at his/her desired speed, there is no free acceleration term. Further, we have $\phi = -20\pi/180 = -0.349$ and the directional strength dependence $w(\phi) = 0.972 > 0$ does not change the direction vector which is given solely by the sum of the unit vectors. Since $\vec{e}_{\vec{x}}$ does not have an y component, the y component of the acceleration is proportional to the y-component of $\vec{e}_{\vec{x}+\Delta\vec{d}}$. We have

$$\vec{e}_{\vec{x}+\Delta\vec{d}} = \frac{\vec{x} + (\vec{v} - \vec{v}_2)\Delta t}{|\vec{x} + (\vec{v} - \vec{v}_2)\Delta t|} = (-0.95, -0.306)$$

confirming the expectation. In fact, we have a negative y component if $v_y - v_{2y} < 0$

Calculating the full interaction acceleration (with $w = 1$) gives

$$-\nabla\Phi^{\text{int}}(\vec{x}) = (-0.226, -0.035)\,\text{m/s}^2$$

(f) We expect now the subject pedestrian to swerve to the left because this pedestrian anticipates that, after the anticipation time, the target pedestrian will be at his/her right side, so we expect a positive y acceleration. Such an anticipation is also contained in the SFM with the presented **elliptical specification II**. A calculation as in (e) gives, indeed,

$$-\nabla\Phi^{\text{int}}(\vec{x}) = (-0.227, 0.034)\,\text{m/s}^2$$

confirming the expectation

Solution to Problem 11.3: Single-file fundamental diagram

(a) The homogeneous steady-state condition implies that the accelerations of all peddestrians i are zero and all pedestrians have the same speed $v_i = v$ and distance $\Delta x_i = d$ from each other,

$$x_{i-1} - x_i = \Delta x_i = d, \quad v_i = V, \quad \frac{\mathrm{d}v_i}{\mathrm{d}t} = 0.$$

For these conditions, the shielded SFM becomes

$$\begin{aligned} 0 = \frac{\mathrm{d}v_i}{\mathrm{d}t} &= \frac{v_0 - V}{\tau} + \sum_{j=-\infty}^{i-1} f_{ij} + \sum_{j'=i+1}^{\infty} f_{ij'} \\ &= \frac{v_0 - V}{\tau} - 1 \sum_{l=1}^{\infty} A e^{-l\Delta x/B} + \lambda \sum_{l'=1}^{\infty} A e^{-l'\Delta x/B} \\ &= \frac{v_0 - V}{\tau} - A e^{-\Delta x/B} + \lambda A e^{-\Delta x/B} \\ &= \frac{v_0 - V}{\tau} - A(1-\lambda) e^{-d/B}, \end{aligned}$$

so we obtain the fundamental speed-distance relation

$$V(d) = v_0 - \tau A(1-\lambda)\, e^{-d/B} \tag{1}$$

(b) With the standing-queue condition $V(d_0) = 0$, we obtain

$$0 = v_0 - \tau A(1-\lambda)\, e^{-d_0/B},$$

so

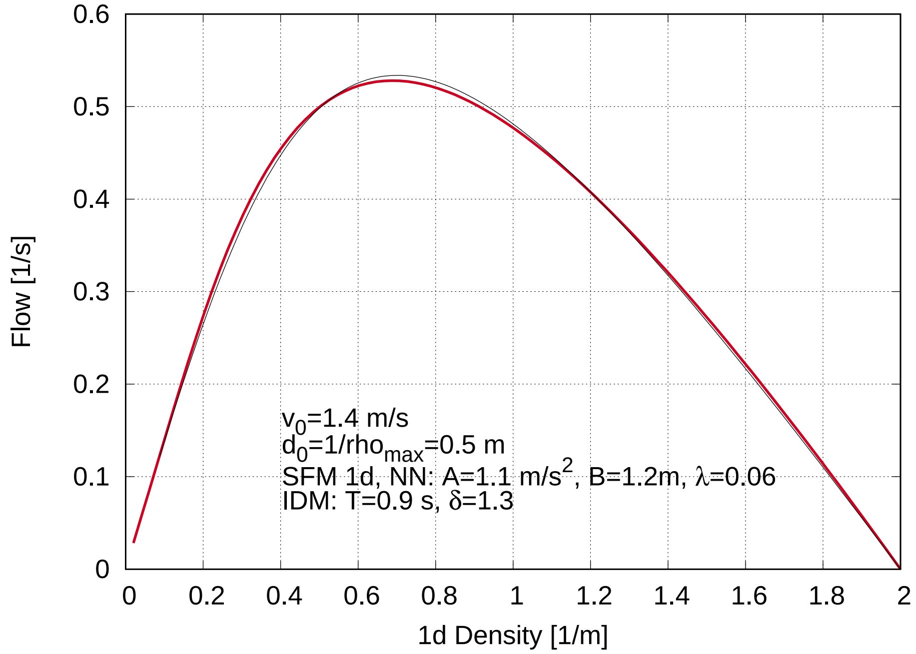
$$A = \frac{v_0}{\tau(1-\lambda)}\, e^{d_0/B}$$

(c) With this condition, the steady-state condition (1) becomes

$$V(d) = v_0 \left[1 - e^{-(d-d_0)/B}\right]$$

and the fundamental diagram (homogeneous steady-state flow-density relation)

$$Q(\rho^{\mathrm{1d}}) = \rho^{\mathrm{1d}} V(1/\rho^{\mathrm{1d}})$$



0.6
0.5
0.4
0.3
0.2
0.1
0
Flow [1/s]
v0=1.4 m/s
d0=1/rhomax=0.5 m
SFM 1d, NN: A=1.1 m/s2, B=1.2m, λ=0.06
IDM: T=0.9 s, δ=1.3
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
1d Density [1/m]