# Traffic Flow Dynamics and Simulation

## Summer semester, Solutions to Work Sheet 7, page 1

### Solution to Problem 7.1: Dissolving queues at a traffic light

When the traffic light turns green, the traffic flow passes the traffic light in the maximum-flow state. For the triangular fundamental diagram, the speed at the maximum-flow state is equal to the desired speed and the transition from the waiting queue (density $\rho_{\text{max}}$) to the maximum-flow state propagates backwards at a velocity $c = -l_{\text{eff}}/T$ corresponding to the congested slope of the fundamental diagram. In the microscopic picture, every follower starts a time interval $T$ later than its leader and instantaneously accelerates to $V_0$. This suggests to interpret $T$ as the reaction time of each driver, so $|c|$ is simply the distance between two queued vehicles divided by the reaction time.

We emphasize, however, that the LWR model does not contain any reaction time. Moreover, the above microscopic interpretation no longer holds for LWR models with other fundamental diagrams. Therefore, another interpretation is more to the point. As above, the driver instantaneously starts from zero to $V_0$ which follows directly from the sharp macroscopic shock fronts. However, the drivers only start their „rocket-like" acceleration when there is enough time headway at $V_0$. Thus, $|c|$ is the distance between two queued vehicles divided by the desired time gap $T$ in car-following mode. Similar considerations apply for concave fundamental diagrams (such as the parabola-shaped of Problem **??**). This allows following general conclusion:

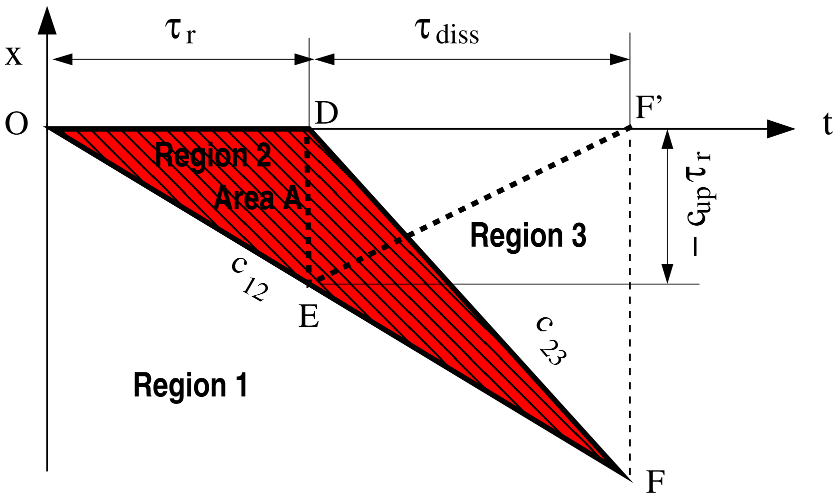
> *The fact that not all drivers start simultaneously at traffic lights is not caused by reaction times but by the higher space requirement of moving with respect to standing vehicles: It simply takes some time for the already started vehicles to make this space.*

### Solution to Problem 7.2: Total waiting time during one red phase of a traffic light

The total waiting time in the queue is equal to the number $n(t)$ of vehicles waiting at a given time, integrated over the duration of the queue: Defining $t = 0$ as the begin of the red phase and $x = 0$ as the position of the stopping line, this means

$$\tau_{\text{tot}} = \int_0^{\tau_r+\tau_{\text{diss}}} n(t)\,\mathrm{d}t = \int_0^{\tau_r+\tau_{\text{diss}}} \int_{x_u(t)}^{x_o(t)} \rho_{\text{max}}\;\mathrm{d}x\;\;\mathrm{d}t = \rho_{\text{max}} A,$$

i.e., the total waiting time is equal to the jam density times the area of the queue in space-time (cf. the following diagram).




The area of the congested area is equal to the sum of the area of the two right-angled triangles with the legs $(\tau_\mathrm{r}, -c_{12}\tau_\mathrm{r})$ and $(\tau_\mathrm{diss}, -c_{12}\tau_\mathrm{r})$, respectively:

$$\tau_\mathrm{tot} = \frac{1}{2}\rho_\mathrm{max}\left(-c_{12}\tau_\mathrm{r}^2 - c_{12}\tau_\mathrm{r}\tau_\mathrm{diss}\right).$$

To obtain the second right-angled triangle DEF’, we have shifted the point F of the original triangle DEF to F’ which does not change the enclosed area. Furthermore, we have the geometrical relation (cf. the figure again)

$$c_{12}\tau_\mathrm{r} = (c_{23} - c_{12})\tau_\mathrm{diss},$$

i.e., $\tau_\mathrm{diss} = c_{12}\tau_\mathrm{r}/(c_{23} - c_{12})$. Inserting this into the expression for $\tau_\mathrm{tot}$ finally gives

$$\tau_\mathrm{tot} = \frac{1}{2}\rho_\mathrm{max}\tau_\mathrm{r}^2\,\frac{c_{12}c_{23}}{c_{12} - c_{23}}$$

with

$$c_{12} = \frac{Q_\mathrm{in}}{Q_\mathrm{in}/V_0 - \rho_\mathrm{max}}, \quad c_{23} = -\frac{1}{\rho_\mathrm{max}T}.$$

The total waiting time ...

- increases with the *square* of the red time,
- is, for small demands, essentially proportional *square* of the demand $Q_\mathrm{in}$,
- diverges if the demand tends to $Q_\mathrm{max}$ (in reality, it diverges if the demand is greater than $Q_\mathrm{max}$ multiplied with the percentage of the greentime during one cycle.)

## Solution to Problem 7.3: Two consecutive signalized intersections: green waves

(a) Capacity:

$$C = Q_{\text{max}} = \frac{1}{T\left(1 + \frac{l_{\text{eff}}}{v_0 T}\right)} = \underline{\underline{0.5\,\text{s}^{-1}}} = \underline{\underline{1\,800\,\text{veh/h}}}.$$

(b) Density of the queues:

$$\rho_{\text{queue}} = \rho_{\text{max}} = \frac{1}{l_{\text{eff}}} = \underline{\underline{133\,\text{veh/km}}}$$

Propagation velocity of perturbations inside the queues:

$$c_{\text{cong}} = -\frac{l_{\text{eff}}}{T} = \underline{\underline{-5\,\text{m/s}}} = \underline{\underline{-18\,\text{km/h}}}$$

This velocity is equal to the slope of the congested branch of the FD and therefore is also valid for the downstream front queue→free (=tip of the FD).

(c) The assumption of a constant demand is only sensible if there are no traffic lights for several kilometers upstream, i.e., for the first traffic light when entering a city. For traffic lights further downstream, the strongly fluctuating inflow is controlled by the lights further upstream.

To estimate the effective capacity over one cycle, we just multiply the free-flow capacity $Q_{\text{max}}$ with the percentage time of the green phases (no yellow/amber phases are mentioned; furthermore, the LWR vehicles accelerate instantaneously):

$$\overline{K} = Q_{\text{max}} \frac{70}{110} = \underline{\underline{1\,145\,\text{veh/h}}}$$

This is sufficient for a demand $Q_{\text{in}} = 1\,029\,\text{veh/h}$

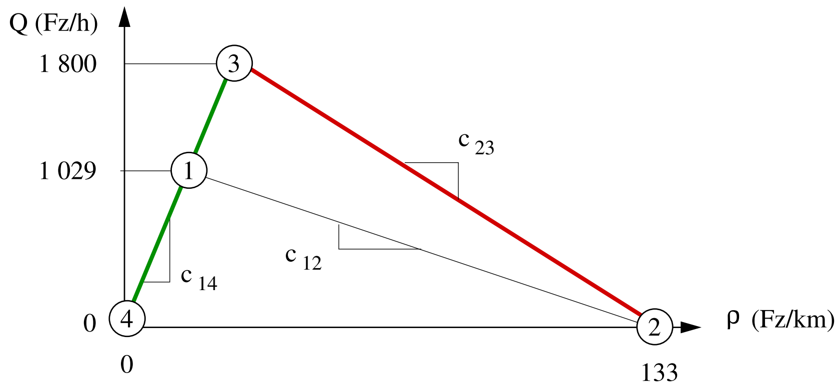
(d) We have following states (cf. the figure):

(i) Inflow: free traffic, $Q_1 = 1\,029\,\text{veh/h}$, $\rho_1 = \frac{Q_1}{V_0} = 19.0\,\text{veh/km}$

(ii) Queue: $Q_2 = 0$, $\rho_2 = \rho_{\text{max}} = 133.3\,\text{veh/km}$

(iii) Outflow from the queue after the light turns green: $Q_3 = Q_{\text{max}} = 1\,800\,\text{veh/h}$, $\rho_3 = \frac{Q_3}{V_0} = 33.3\,\text{veh/km}$

(iv) Empty road downstream of a red traffic light: $Q_4 = 0$, $\rho_4 = 0$.



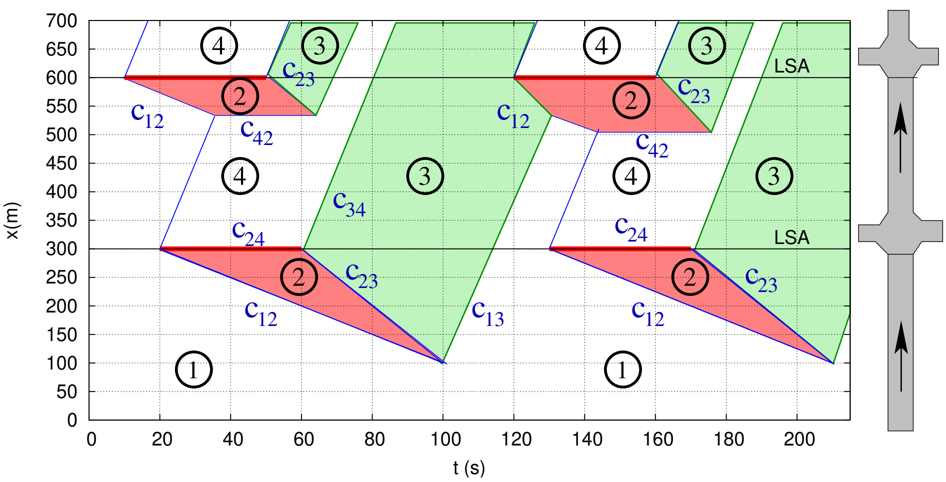
From the *shockwave formula*, this gives following front propagation velocities (for the notation cf. the figure):

$$
\begin{aligned}
c_{12} = c_{21} &= \frac{Q_2 - Q_1}{\rho_2 - \rho_1} = -2.5\,\mathrm{m/s} = -9\,\mathrm{km/h}, \\
c_{13} = c_{31} = c_{14} = c_{41} = c_{34} = c_{43} &= \frac{Q_3 - Q_1}{\rho_3 - \rho_1} = 15\,\mathrm{m/s} = 54\,\mathrm{km/h}, \\
c_{23} = c_{32} &= \frac{Q_2 - Q_3}{\rho_2 - \rho_3} = c_{\mathrm{cong}} = -5\,\mathrm{m/s} = -18\,\mathrm{km/h},
\end{aligned}
$$

For the drawing, we have:

- Inflow-queue: one negative vertical box edge per one horizontal edge,
- Outflow: Two negative vertical box edges per horizontal edge
- Empty road - outflow and vice versa: six positiv vertical edges per horizontal edge

(e) Sketch:




## Solution to Problem 7.4: Flow instability in Payne's model and in the Kerner-Konhäuser Model

**For both models, (a) and (b):** According to the problem setting, traffic flow is stable for the Payne and Kerner-Konhäuser models if the *stability* condition

$$\left(\rho V_{\mathrm{e}}'(\rho)\right)^2 \leq P'(\rho).$$

applies. For the triangular fundamental diagram as specified in the problem formulation, the gradient of the speed-density relation reads

$$V_e'(\rho) = \begin{cases} 0 & \rho \le \rho_\text{C}, \\ -\frac{1}{\rho^2 T} & \rho > \rho_\text{C}, \end{cases}$$

with the density at capacity $\rho_\text{C} = 1/(v_0 T + l_\text{eff}) = 20\,\text{vehicles/km}$. For free traffic ($\rho < \rho_\text{C}$), there are no interactions in both models ($V_e'(\rho) = 0$) and therefore we have unconditional stability.

According to the above stability condition, congested traffic flow ($\rho \ge \rho_\text{C}$) is stable if

$$\frac{1}{(\rho T)^2} \le P'(\rho)$$

The derivative $P'(\rho)$ of the pressure term $P(\rho)$ of the two models can be found by comparing the model's gradient term with the partial derivative of the respective model equation (put on the right-hand side) with the expression $-\frac{1}{\rho}\frac{\partial P(\rho(x))}{\partial x} = -\frac{1}{\rho}P'(\rho)\frac{\partial \rho}{\partial x}$ defining the pressure. For Payne's model, this gives

$$\begin{aligned} -\frac{1}{\rho}P'(\rho)\frac{\partial \rho}{\partial x} &= \frac{V_e'(\rho)}{2\rho\tau}\frac{\partial \rho}{\partial x} \\ P'(\rho) &= \frac{-V_e'(\rho)}{2\tau} \end{aligned}$$

and for the KK model simply

$$P'(\rho) = \theta_0$$

(a) Inserting this into the stability condition gives for Payne's model

$$\begin{aligned} \frac{1}{(\rho T)^2} &\le P'(\rho) \\ \frac{1}{(\rho T)^2} &\le -\frac{V_e'(\rho)}{2\tau} \\ \frac{1}{(\rho T)^2} &\le \frac{1}{2\rho^2 T\tau} \\ \frac{1}{T} &\le \frac{1}{2\tau} \\ \tau &\le T/2 \end{aligned}$$

In summary, Payne's model is stable if either $\rho < \rho_c$ (free) or, if congested, $\tau \le T/2$

(b) For the KK model, we have

$$\frac{1}{(\rho T)^2} \le \theta_0 \;\Rightarrow\; \rho^2 > \frac{1}{\theta_0 T^2}$$

So, the KK model is (flow) stable if either $\rho < \rho_c$ (free) or, if congested, $\rho > 1/(T\sqrt{\theta_0})$ or, (cf the figure)

$$(\rho < \rho_\text{c}) \text{ OR } \left(\rho > \frac{1}{T\sqrt{\theta_0}}\right).$$

Hence, if the stability limit should be at $\rho_{c4} = 50\,/\mathrm{km}$, we have for $T = 1.1\,\mathrm{s}$ (cf the figure)

$$\theta_0 = \frac{1}{\rho_c^2 T^2} = 331 \frac{\mathrm{m}^2}{\mathrm{s}^2}.$$